A Survey of Absolute p-adic Anabelian Geometry

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§1. Absolute Anabelian Geometry

Let F_1, F_2 be fields, with absolute Galois groups G_{F_1}, G_{F_2};

$$\phi : G_{F_1} \xrightarrow{\sim} G_{F_2}$$

an isomorphism of profinite groups. Then:

QUESTION:
Does ϕ necessarily arise from an isomorphism of fields $F_1 \xrightarrow{\sim} F_2$?

ANSWERS:

· YES, if F_i are number fields (NF), by Neukirch-Uchida (NU).

· NO, if F_i are p-adic local fields (pLF).

· YES, if F_i are pLF, and ϕ preserves the ramification filtration, or, alternatively, the isom. class of the topological Galois module "$\mathbb{C}_p$" (cf. [$\mathbb{Q}_p$GC]).

Thus, NO/YES for pLF is a measure of the extent to which ϕ preserves the respective "p-adic Hodge theories" (pHT) of the F_i — as if pHT is a sort of "holomorphic structure" on an underlying "real analytic/ topological manifold" G_{F_i}.

Now let:

$\mathbb{V}$ be a class of varieties;

$\mathbb{F}$ a class of fields.

If $V \in \mathbb{V}$, $F \in \mathbb{F}$, write:

$\Pi_V \stackrel{\text{def}}{=} \pi_1(V)$ (étale fund. group);

$G_F \stackrel{\text{def}}{=} G_F$ (absolute Galois group).

Consider the following assertions:

(rel $\mathbb{V}\mathbb{F}$GC) For $V_i \in \mathbb{V}$ (where $i = 1, 2$) over $F \in \mathbb{F}$, the natural map

$$\text{Isom}_F(V_1, V_2) \to \text{OutIsom}_{G_F}(\Pi_{V_1}, \Pi_{V_2})$$

is a bijection.

(abs $\mathbb{VF}$GC) For $V_i \in \mathbb{V}$ over $F_i \in \mathbb{F}$ (where $i = 1, 2$), the natural map

$$\mathrm{Isom}(V_1, V_2) \to \mathrm{OutIsom}(\Pi_{V_1}, \Pi_{V_2})$$

is a bijection.

When $\mathbb{V}$ = "hyperbolic curves", write:

pGC $\stackrel{\mathrm{def}}{=}$ $\mathbb{VF}$GC, when $\mathbb{F}$ = "pLF";

NFGC $\stackrel{\mathrm{def}}{=}$ $\mathbb{VF}$GC, when $\mathbb{F}$ = "NF".

Thus, by "YES for NF" (NU), we have:

(rel NFGC) $\iff$ (abs NFGC)

By contrast, even though (rel pGC) is known (cf. [pGC]), (abs pGC) is only known in certain special cases, to be discussed in the present survey.

As discussed above, (abs pGC) involves the subtle issue of preserving the "pHT", i.e., the "holomorphic structure", on G_K, for $K/\mathbb{Q}_p < \infty$.

Motivation for (abs pGC): work on ABC Conjecture, in particular,

"Inter-universal Teichmüller Theory"
(IUTeich)

(work in progress).

Idea: construct "canonical Teich. lifts" of pLF, NF, i.e.:

scheme theory $\longleftrightarrow$ char. p scheme theory
IUTeich lifts $\longleftrightarrow$ p-adic Witt/Teich. lifts

Put another way, trying to construct a sort of

$$\text{“}\mathbb{Z}_p \times_{\mathbb{F}_1} \mathbb{Z}_p\text{”}$$

where:

one $\mathbb{Z}_p$ is scheme-theoretic,
the other $\mathbb{Z}_p$ is Galois-theoretic.

Then (abs pGC) arises in developing the theory of the "Galois-theoretic $\mathbb{Z}_p$".

§2. Canonical Curves

Let

$$\mathbb{V} \stackrel{\text{def}}{=} \text{“hyperbolic curves”}$$

$$\mathbb{F} \stackrel{\text{def}}{=} \text{“}p\text{LF”}$$

In p-adic Teichmüller theory (cf. Serre-Tate theory; Bers uniformizations over $\mathbb{C}$), one has a notion of canonical liftings of certain hyperbolic curves over finite fields (equipped with certain auxiliary data) to rings of Witt vectors of the base fields. Thus, we also consider (when $p \geq 3$):

$$\mathbb{V}^{\text{can}} \stackrel{\text{def}}{=} \text{“can. lifted hyperbolic curves”}$$

$$\mathbb{F}^{\text{can}} \stackrel{\text{def}}{=} \text{“absolutely unramified }p\text{LF”}$$

Thus, if we fix the “type (g, r)”, then the resulting set of isomorphism classes of $\mathbb{V}^{\text{can}}$ is countably infinite and Zariski dense in the moduli stack of hyperbolic curves of type (g, r).

In the case of canonical curves, we have a somewhat weaker result than the full "(abs pGC)", which was, in fact, the first result obtained (by the lecturer) in absolute p-adic anabelian geometry (cf. [Canon]):

Theorem: Let $X_1, X_2 \in \mathbb{V}$,

$$\phi : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$$

an isomorphism of profinite groups. Then:

(i) $X_1 \in \mathbb{V}^{\mathrm{can}} \iff X_2 \in \mathbb{V}^{\mathrm{can}}$.

(ii) Suppose that X_1 or X_2 belongs to $\mathbb{V}^{\mathrm{can}}$. Then ϕ induces a functorial isomorphism of the respective log special fibers of X_1, X_2, which is, moreover, compatible with the canonical deformations of these log special fibers constituted by X_1, X_2.

§3. Curves with Belyi Maps

Consider the following ("quasi-Belyi-ness") condition on an (affine) hyperbolic curve X over a field F of char. 0:

(QB) There exist a dominant morphism

$$Y \to (\mathbb{P}^1 \backslash 01\infty)_F,$$

where Y is a hyperbolic curve, together with a finite étale morphism $Y \to X$.

Also, we consider the condition:

(NFQB) $\stackrel{\text{def}}{=}$ (QB) + (X is defined over a NF).

If Z is a proper hyperbolic curve of genus ≥ 2 over F, $r > 0$, then

$$Z \setminus (\text{generic } r\text{-tuple of points})$$

is not (QB) (A. Tamagawa — cf. [Config]).

The first result obtained (by the lecturer) concerning (abs pGC) is the following (cf. [Cusp]):

Theorem A: (abs $\mathbb{V}\mathbb{F}$GC) holds, for

$$\mathbb{V} \stackrel{\text{def}}{=} \text{(NFQB)-curves}, \ \mathbb{F} \stackrel{\text{def}}{=} \text{“}p\text{LF”}.$$

Subsequent to this result, A. Tamagawa refined the technique of “applying Belyi maps to prove (abs pGC)” via the following result, which is of independent interest:

Theorem* B: Every Lubin-Tate group appears as a subquotient of the p-adic Tate module of some abelian variety over a NF.

Thm. B allows one to prove the following generalization of Thm. A:

Theorem* C: (abs $\mathbb{V}\mathbb{F}$GC) holds, for

$$\mathbb{V} \stackrel{\text{def}}{=} \text{(QB)-curves}, \ \mathbb{F} \stackrel{\text{def}}{=} \text{“}p\text{LF”}.$$

* orally communicated, unwritten as of the time of this lecture

Remarks:

· Although Thm. A is weaker than Thm. C, the technique of Thm. A is "NF-friendly", hence yields a new proof of (abs NFGC) for (NFQB)-curves that does not rely on NU! This is the first example of a proof of (a certain consequence of) NU that involves an explicit construction of the NF.

· Thm. C is the first version of (abs pGC) that applies to uncountably many curves, as well as to arbitrary multiply-punctured elliptic curves.

· It appears likely (?) that Thm. C may be generalized to a "Hom-version" (i.e., for open homomorphisms, as opposed to isomorphisms, of arith. fund. groups).

§4. Configuration Spaces

Let X be a hyperbolic curve, $n \geq 1$ an integer. Then consider the n-th configuration space

$$X \times \ldots \times X \backslash \text{diagonals}$$

(where the product is of n copies of X) associated to X.

Theorem: Let $\mathbb{F} \stackrel{\text{def}}{=}$ "pLF";

$$\mathbb{V}$$

the class of n-th configuration spaces associated to hyperbolic curves of type

$$(g, r) \neq (0, 3); (1, 1),$$

where

$$n \geq 3 \text{ if } r = 0 \text{ (the proper case)},$$

$$n \geq 2 \text{ if } r > 0 \text{ (the affine case)}.$$

Then (abs $\mathbb{VF}$GC) holds.

Proof: Combine joint work with A. Tamagawa (cf. [Config]) on the geometry of configuration spaces, with a certain "combinatorial version of the GC" (cf. [CombGC]), and the (abs pGC) applied to the copy of $\mathbb{P}^1 \backslash 01\infty$ "lying inside the boundary of the configuration space" (cf. the assumption on n). $\square$

Note that this is the first result of absolute p-adic anabelian geometry that applies to arbitrary hyperbolic curves.

§5. Further Directions

Let

$$\mathbb{V} \stackrel{\mathrm{def}}{=} \text{hyperbolic curves},$$

$$\mathbb{F} \stackrel{\mathrm{def}}{=} p\mathrm{LF}.$$

If Σ is a set of primes, $X \in \mathbb{V}$, write

$$\Pi_X^\Sigma$$

for the max. geometrically pro-Σ quotient of Π_X.

Then earlier this year, the lecturer showed the following:

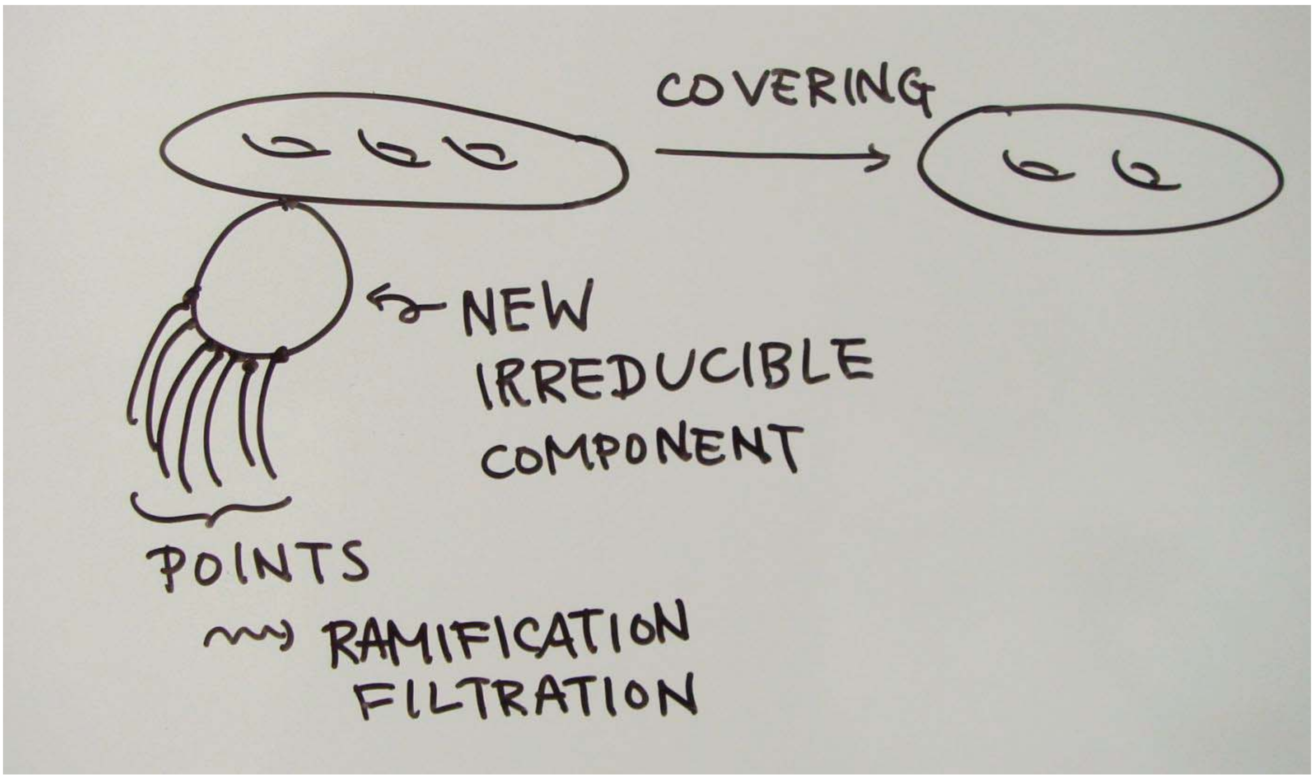
Theorem*: Let $X_1, X_2 \in \mathbb{V}$,

$$\phi : \Pi_{X_1}^\Sigma \xrightarrow{\sim} \Pi_{X_2}^\Sigma$$

an isomorphism of profinite groups. Suppose that $p, l \in \Sigma$, where $l \neq p$. Then ϕ arises geometrically if and only if ϕ is point-theoretic (i.e., preserves decomposition groups of closed points of X_1, X_2).

* unwritten as of the time of this lecture

Proof: If $X \in \mathbb{V}$ lies over $K \in \mathbb{F}$, then by considering various finite étale coverings of X of order a power of p, one may effect arbitrarily many "blow-ups". Then careful inspection of the collection of closed points contained in the interior of the "new irreducible components" arising from these "blow-ups" shows that such collections of points correspond essentially to, i.e., may be thought of as "geometric realizations" of, various portions of the ramification filtration of G_K. Thus, one concludes via the theory of [$\mathbb{Q}_p$GC], together with the (rel pGC). $\square$



Thus, it remains to show point-theoreticity. It appears likely that this should be possible if one can answer the following question in the affirmative:

QUESTION: In the notation of the Theorem, write $K_i \in \mathbb{F}$ for the base field of X_i; $\Delta_i^\Sigma \stackrel{\text{def}}{=} \text{Ker}(\Pi_i^\Sigma \twoheadrightarrow G_{K_i})$. Let (for $i = 1, 2$)

$$H_i \subseteq \Delta_i^\Sigma$$

be an open subgroup such that $\phi(H_1) = H_2$. Then does the natural isomorphism (induced by ϕ)

$$H^1(G_{K_1}, H_1^{\text{ab}} \otimes \mathbb{Z}_p) \xrightarrow{\sim} H^1(G_{K_2}, H_2^{\text{ab}} \otimes \mathbb{Z}_p)$$

preserve "$H^1_{\text{f}} \subseteq H^1$"?

Put another way, does the resulting isomorphism $G_{K_1} \xrightarrow{\sim} G_{K_2}$ preserve Hodge-Tate decompositions of Galois modules which are known to be Hodge-Tate for both G_{K_1}, G_{K_2}?

Remarks:

· The question may (easily) be answered in the affirmative when the Jacobians of the coverings determined by the H_i are ordinary.

· This question seems to be interesting as a question in p-adic Hodge theory, independent of anabelian geometry.